Feb. 5, 2014

# Related Rates

(we pretend balloons are spherical, even if they aren't)

Consider: Pumping air into a balloon

- fairly easy to measure the rate at which the balloon is being filled
  this is the rate of change of volume with respect to time.
- What about measuring the rate of increase of the radius?

It's for situations like these that we have a problem solving technique that we call related rates.

Ex:(1) Pump air into a spherical balloon so its volume increases at a rate of 100 $cm^3/s$.
How fast is the radius of the balloon increasing when the diameter is 50 cm?

Analyze the Problem:

Info we have: rate of inc. of volume is 100 $cm^3/s$

we want: rate of inc. of radius when diameter is 50 cm.

## RATES OF CHANGE ARE DERIVATIVES

the suggestive notation:

$t$ = time

$V$ = volume (a func of time)

$r$ = radius (also a func of time)

$\frac{dV}{dt}$ = rate of inc of volume with respect to time

$\frac{dr}{dt}$ = rate of inc of radius with respect to time

Rewrite what we have: $\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

Unknown: $\frac{dr}{dt}$ when $r = 25$ cm

How do we get this?

* Write an equation that relates V and r
  - Volume of a sphere: $\frac{4}{3}\pi r^3 = V$

* Take derivative of both sides (implicit)
  Remember what your variables depend on —
  you will need the chain rule

We diff. with respect to t in many problems.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$1 \cdot \frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \underbrace{\frac{dr}{dt}}_{\text{want this when } r = 25 \text{ cm}}$$

Plug in $\frac{dV}{dt} = 100$, $r = 25$

$$100 = 4\pi(25)^2 \frac{dr}{dt}$$ solve for $\frac{dr}{dt}$

$$\frac{100}{4\pi(25)^2} = \frac{dr}{dt}$$

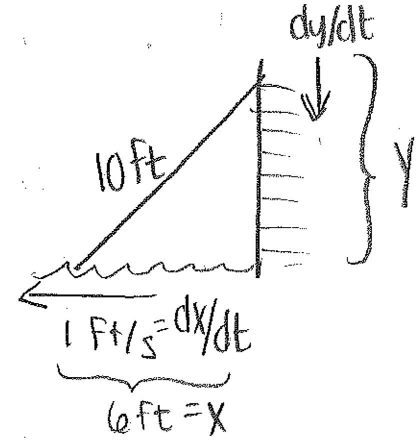
$$\boxed{\frac{1}{25\pi} = \frac{dr}{dt}}$$

Conclusion: the radius is increasing at a rate of $1/(25\pi) \approx 0.0127$ cm/s

Example: (2) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Steps: (1) Read Problem Carefully

(2) Draw a diagram when possible




(3) Introduce notation. Assign symbols to quantities that are funcs of time. (sometimes they are funcs not of time. Ex later)

(4) Write an equation relating variables

$$10^2 = x^2 + y^2$$
$$100 = x^2 + y^2$$

(5) Use Chain Rule to diffentiate both sides with respect to desired variable (usually time)

$$\frac{d}{dt}(100) = \frac{d}{dt}(x^2 + y^2)$$

$$0 = 2x \cdot \frac{dx}{dt} + 2y \frac{dy}{dt}$$

(6) Substitute what you have and solve for what you want

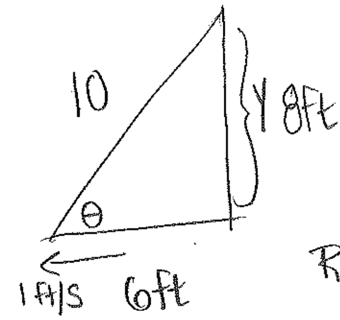
$$0 = 2(6)(1) + 2y \frac{dy}{dt}$$

we can find y: $y = \sqrt{100 - 36} = 8$

$$0 = 12 + 16 \cdot \frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{-3}{4} \text{ ft/s}$$

We can calculates related rates w/ respect to other values.

What is the rate of change in the angle between ladder and ground with respect to the distance from the top of the ladder and the ground in this situation?




want $\frac{d\theta}{dy}$ ← change in angle, ← change in height

Relate θ and y:

$$\sin\theta = \frac{y}{10} \qquad \left(\sin\theta = \frac{opp}{hyp}\right)$$

* differentiate with respect to y:

$$\frac{d}{dy}(\sin\theta) = \frac{d}{dy}\left(\frac{y}{10}\right)$$

$$\underbrace{\cos\theta}_{=6/10} \cdot \frac{d\theta}{dy} = \frac{1}{10}$$

$$\frac{d\theta}{dy} = \frac{1}{6} \text{ radians/ft}$$

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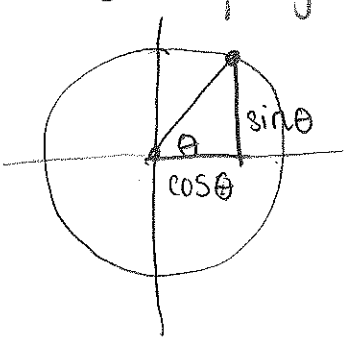
What is the rate of change of the cosine of an angle w/ respect to the sine of an angle when the angle is changing in radians per second at a constant rate?




$$\sin^2\theta + \cos^2\theta = 1$$ differentiate w.r.t. sin θ

$$\frac{d}{\sin\theta}(\sin^2\theta + \cos^2\theta) = 0$$

$$2\sin\theta + 2\cos\theta \cdot \frac{d\cos\theta}{d\sin\theta} = 0 \Leftrightarrow \frac{d\cos\theta}{d\sin\theta} = \frac{-\sin\theta}{\cos\theta}$$